5 Examples

Do as many as you can!

Problem 1. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$ and $f_y(3, 2)$.

$$f_x(3,2) = \lim_{h\to 0} \frac{f(3+h,2)-f(3,2)}{h}$$

$$h=0.5: \frac{f(3.5,2)-f(3,2)}{0.5} = 3.8$$

$$h=-0.5: \frac{f(2.5,2)-f(3,2)}{-0.5} = 11.2$$

$$\Rightarrow f_x(3,2) \approx \frac{3.8+11.2}{2} = 7.5$$

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

$$f_y(3,2) = \lim_{h\to 0} \frac{f(3,2+h)-f(3,2)}{h}$$

$$h=0.2: \frac{f(3,2.2)-f(3,2)}{0.2} = -8$$

$$h=-0.2: \frac{f(3,1.8)-f(3,2)}{-0.2} = -3$$

$$\Rightarrow f_y(3,2) \approx \frac{-8-3}{2} = -5.5$$

Problem 2. Consider the level curves given in Example 2. Determine whether the following partial derivatives are positive or negative at the point P.

a. f_{xx} > 0

b. f_{yy} > 0

c. f_{xy} < 0

Problem 3. Let $f(x, y) = \arctan(y/x)$. Find $f_x(2, 3)$.

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} \quad \Rightarrow f_x(2,3) = -\frac{3}{4+9}$$

$$= -\frac{3}{13}$$

Problem 4. Let $f(x, y, z) = \dfrac{y}{x + y + z}$. Find $f_y(2, 1, -1)$.

(Partial derivatives of functions of 3 or more variables are found the same way: regard all but one variable as constant, and take the derivative with respect to the remaining variable.)

$$f_y(x,y,z) = \frac{(x+y+z)(1)-(y)(1)}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2}$$

$$\Rightarrow f_y(2,1,-1) = \frac{1}{4}$$

Problem 5. Let $f(x,y,z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$. Find $f_x(0,0,\pi/4)$.

$$f_x(x,y,z) = \frac{1}{2}\left(\sin^2 x + \sin^2 y + \sin^2 z\right)^{-\frac{1}{2}}\left(2\sin x \cos x\right)$$

$$= \sin x \cos x \left(\sin^2 x + \sin^2 y + \sin^2 z\right)^{-\frac{1}{2}}$$

$$\Rightarrow f_x\left(0,0,\tfrac{\pi}{4}\right) = 0$$

Problem 6. Find all the second partial derivatives of $f(x,y) = x^4 y - 2x^3 y^2$.

$$f_x(x,y) = 4x^3 y - 6x^2 y^2 \qquad f_y(x,y) = x^4 - 4x^3 y$$

$$\Rightarrow f_{xx}(x,y) = 12x^2 y - 12xy^2 \qquad f_{yy}(x,y) = -4x^3$$

$$f_{xy}(x,y) = f_{yx}(x,y) = 4x^3 - 12x^2 y$$

Problem 7. Let $f(x,y) = \cos(x^2 y)$. Verify that Clairaut's theorem holds: $f_{xy} = f_{yx}$.

$$f_x(x,y) = -\sin(x^2 y)(2xy) \qquad f_y(x,y) = -\sin(x^2 y)(x^2)$$

$$\Rightarrow f_{xy}(x,y) = -\sin(x^2 y)(2x) + (2xy)\left(-\cos(x^2 y)(x^2)\right) = -2x\sin(x^2 y) - 2x^3 y\cos(x^2 y)$$

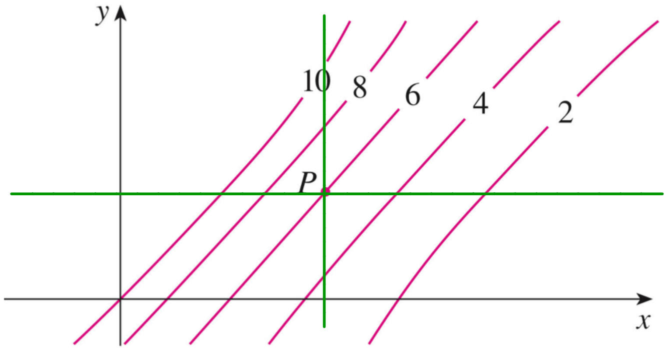
$$f_{yx}(x,y) = -\sin(x^2 y)(2x) + (x^2)\left(-\cos(x^2 y)(2xy)\right) = -2x\sin(x^2 y) - 2x^3 y\cos(x^2 y)$$

Problem 8. Let $f(x,y) = \sin(2x + 5y)$. Find f_{yxy}.

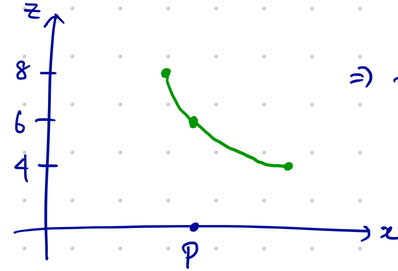
$$f_y = 5\cos(2x + 5y)$$

$$f_{yx} = -10\sin(2x + 5y)$$

$$f_{yxy} = -50\cos(2x + 5y)$$

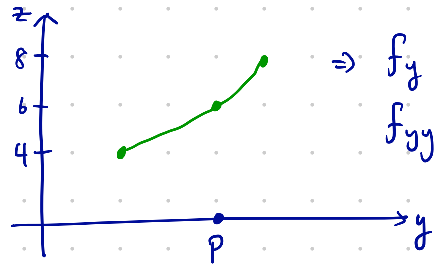


$f_x, f_y, f_{xx}, f_{yy}, f_{xy}$?



$\Rightarrow f_x < 0$

$f_{xx} > 0$



$\Rightarrow f_y > 0$

$f_{yy} > 0$

f_{xy}: how does f_x change as y increases?

contours farther apart in x-direction below P $\Rightarrow$ f_x is less negative

contours closer together in x-direction above P $\Rightarrow$ f_x is more negative

$\Rightarrow f_{xy} < 0$